Intermittency in chaotic rotations

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We examine the rotational dynamics associated with bounded chaotic flows, such as those on chaotic attractors, and find that the dynamics typically exhibits on-off intermittency. In particular, a properly defined chaotic rotation tends to follow, approximately, the phase-space rotation of a harmonic oscillator with occasional bursts away from this nearly uniform rotation. The intermittent behavior is identified in several well studied chaotic systems, and an argument is provided for the generality of this behavior.

Rotation is a fundamental characteristic associated with many natural and technological processes that are mathematically described by a set of coupled first-order autonomous differential equations (flows). Consider, for example, the dynamics on a chaotic attractor. While having a sensitive dependence on initial conditions, trajectories on the attractor are recurrent because the attractor is bounded in the phase space. That is, a chaotic trajectory starting from a point in the attractor must return to the neighborhood of this point infinitely often but never exactly passes through the initial point again, a behavior that resembles rotation. A chaotic rotation can, however, be quite complicated in the sense that there is usually a lack of a well defined center of rotation. Despite this difficulty, there has been an interest in the study of chaotic rotations in the context of phase synchronization [1,2]. There has also been effort to study methods to define proper rotations for general chaotic flows [3].

In this Rapid Communication, we address one question that is fundamental to understanding the rotational structure of chaotic flows: what are the dynamical characteristics of a chaotic rotation with respect to simple rotations such as those produced by harmonic oscillators? Our motivation comes from the intuition and observation that, on average, a chaotic rotation tends to follow a uniform one. In particular, say we consider the phase (angle) variable $\phi(t)$ associated with a chaotic rotation. When having a well defined center, the rotation is proper in the sense that $\phi(t)$ increases monotonically in time. In fact, a linear fit can be found for $\phi(t)$ in large time scales, which indicates that on average, $\phi(t)$ increases linearly with time, the determining characteristic of a uniform rotation. Our key idea is that the average uniform rotation can in fact be regarded as an *invariant property* of the underlying chaotic flow. Thus, a new dynamical system can be defined so that the uniform rotation represents an invariant manifold. A chaotic rotation thus corresponds to, in the new phase space, a trajectory that evolves near the invariant manifold. We find that, such a trajectory exhibits on-off intermittency, a recent subject that has been under extensive study [4]. Our understanding of the rotational structure of chaotic flows is then as follows: a chaotic rotation tends to stay near a uniform rotation with occasional deviation away from it in the course of time evolution. We note that in previous studies, a necessary condition for generating on-off intermittency is that the differential equations describing the system possess a symmetric invariant subspace [4]. The results of this paper imply, however, that on-off intermittency may be more prevalent than previously thought, as it can occur in almost any chaotic flows, regardless of whether the system has a symmetry or not.

We begin by presenting numerical results with the Rössler oscillator [5]: $dx/dt = -\omega_0 y - z$, $dy/dt = \omega_0 x + ay$, and $dz/dt = b + (x - c)z$, where ω_0, a, b, and c are parameters. For the Rössler system, the invariant rotational structure is apparent: the terms $-\omega_0 y$ and $\omega_0 x$ in the x and y equations, respectively, describe the dynamics of a harmonic oscillator of intrinsic frequency ω_0. For many parameter values, the chaotic attractors of the Rössler system possess a well-defined center of rotation, as shown in Fig. 1(a), the plot of $x(t)$ versus $y(t)$, where the parameter values are $\omega_0 = 1.0$, $a = 0.165$, $b = 0.2$, and $c = 10.0$. The amplitude of the rotation can be defined as $r(t) = \sqrt{x^2(t) + y^2(t)}$, and the phase variable $\phi(t)$ is $\phi(t) = \tan^{-1}[y(t)/x(t)]$. Figures 1(b) and 1(c) show the time traces of $r(t)$ and $\phi(t)$, respectively, where
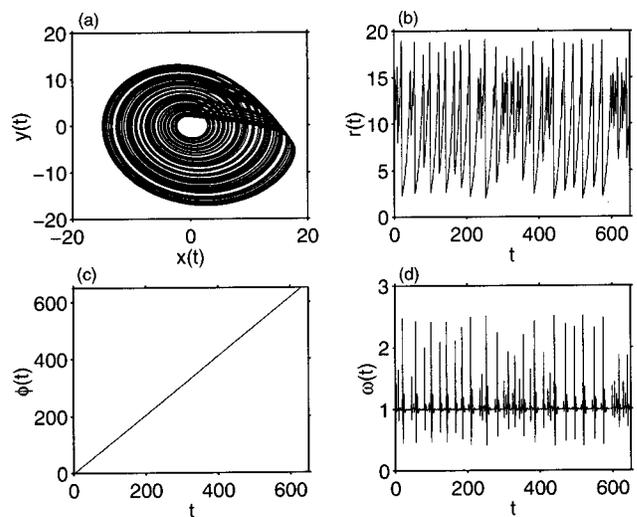

FIG. 1. For the Rössler system: (a) a typical chaotic rotation in the (x,y) plane, (b) amplitude $r(t)$ of the rotation, (c) phase angle $\phi(t)$ of the rotation, and (d) time trace of the instantaneous frequency $\omega(t)$. Apparently, $\omega(t)$ exhibits an on-off intermittent behavior.

we see that the amplitude of the rotation is apparently chaotic (hence the term ''chaotic rotation''), and the phase variable $\phi(t)$ appears to increase linearly with slope ω_0, indicating that on average, we have $\langle\phi(T)\rangle=\omega_0 t$. However, $\phi(t)$ exhibits fluctuations about $\omega_0 t$. Since the underlying harmonic oscillator has a constant frequency, which can be regarded as a natural invariant property, we choose the instantaneous frequency $\omega(t)\equiv d\phi(t)/dt$ to be the phase space variable in the new dynamical system. Figure 1(d) shows the time trace of $\omega(t)$, from which we see that indeed, the instantaneous frequency of the chaotic rotation exhibits a behavior that resembles on-off intermittency.

The reason that $\omega(t)$ in Fig. 1(d) exhibits an on-off intermittent type of behavior can be seen as follows. Because the rotation in Fig. 1(a) is well defined, we can rewrite the Rössler equations in the cylindrical coordinate (r,ϕ,z): $dr/dt=ar\sin^2\phi-z\cos\phi$, $d\phi/dt=\omega_0+\frac{1}{2}a\sin 2\phi+(z/r)\sin\phi$, and $dz/dt=b-cz+rz\cos\phi$. Taking the derivative of $d\phi/dt$ yields the equation governing the dynamics of the instantaneous frequency $\omega(t)$. We obtain, after change of variable $\Omega(t)=\omega(t)-\omega_0$, the following equation:

$$\frac{d\Omega}{dt}=\alpha(t)\Omega+\beta(t),\qquad(1)$$

where $\alpha(t)$ and $\beta(t)$ are given by $\alpha(t)=a\cos 2\phi+(z/r)\cos\phi$, and $\beta(t)=(1/r^2)[r(b-cz+rz\cos\phi)-z(ar\sin^2\phi-z\cos\phi]\sin\phi$. The key observations are (1) the terms $\alpha(t)$ and $\beta(t)$ are random in large time scales because the variables $r(t)$, $z(t)$, and $\phi(t)$ are chaotic, and (2) $\alpha(t)$ and $\beta(t)$ are zero-mean random variables because on average, the frequency of the rotation cannot change, as stipulated by the physical constraint that the flow is bounded. As such, we see that Eq. (1) resembles the general model used to describe on-off intermittency in the presence of additive noise [6]. In particular, if the additive noise term $\beta(t)$ is absent, the dynamical variable $\Omega(t)$ possesses an invariant subspace: $\Omega=0$ (or $\omega=\omega_0$). Thus, it is reasonable that the one-dimensional dynamical system Eq. (1) constructed from the original three-dimensional Rössler system exhibits on-off intermittency.

We note that the ''off'' state in Fig. 1(d) is rather sharp: $\omega\approx\omega_0$. The reason is that the backbone of the Rössler system is an ideal harmonic oscillator of frequency ω_0. For general chaotic flows for which the equations do not apparently contain these of a harmonic oscillator, we expect the ''off'' state to be broadened. This can be seen by considering another well studied chaotic flow: the Lorenz system [7]. The classical Lorenz equations are $dx/dt=10(y-x)$, $dy/dt=28x-y-xz$, and $dz/dt=-(\frac{8}{3})z+xy$, for which there is a chaotic attractor [7]. The attractor apparently has two centers of rotation and, hence, it is not obvious how a meaningful phase variable can be defined. However, it was suggested in Ref. [2] that the following variable: $u(t)=\sqrt{x^2(t)+y^2(t)}$ corresponds to a proper rotation. In particular, it can be easily verified that for the time series $u(t)$, in a large time interval, the number of zero-crossing points is the same as the number of local extrema, which is a defining aspect of a proper rotation. The phase variable in this case can then be obtained as follows. One first constructs the Hilbert transform of $u(t)$,
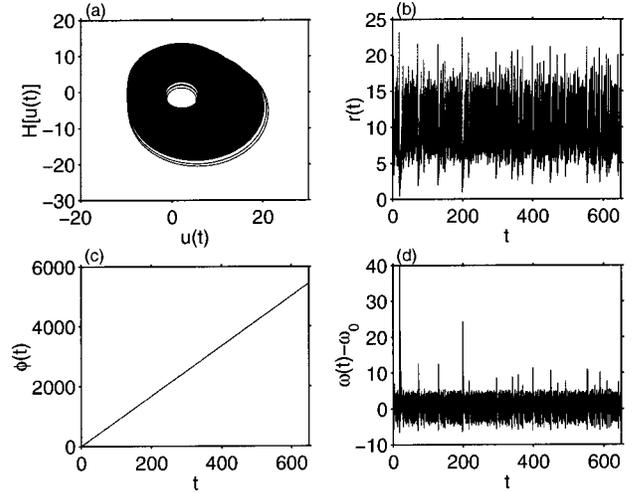

FIG. 2. For the Lorenz system: (a) a typical chaotic rotation in the $[u,H(u)]$ plane, (b) amplitude $r(t)$ of the rotation, (c) phase angle $\phi(t)$ of the rotation, and (d) time trace of the instantaneous frequency $\omega(t)$.

$$H[u(t)]=\mathrm{PV}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t')}{t-t'}dt'\right),\qquad(2)$$

where PV stands for the Cauchy principal value of the integral. An analytic signal [8] $\psi_u(t)$ can then be constructed, $\psi_u(t)=u(t)+iH[u(t)]$, from which the amplitude and the phase of the rotation can be defined: $\psi_u(t)=r(t)\exp[i\phi(t)]$. Figure 2(a) show the chaotic rotation in the complex plane of the analytic signal ψ_u for the Lorenz chaotic attractor. The rotation is apparently proper because it has a unique center [9]. Figures 2(b) and 2(c) show the amplitude $r(t)$ and the phase $\phi(t)$ of the rotation, respectively. As in Figs. 1(b) and 1(c), we see that $r(t)$ is chaotic and $\phi(t)$ is monotonic; two defining characteristics of a chaotic rotation. Figure 2(d) shows the instantaneous frequency $\omega(t)$ of the rotation about the average frequency ω_0, which exhibits an on-off intermittent behavior. The ''off'' state in Fig. 2(d) is, however, broadened compared with that in Fig. 1(d), indicating that the chaotic rotation of the Lorenz attractor has a range of instantaneous frequencies near the ''off'' state.

We now present an argument for the intermittent dynamics of the instantaneous frequencies in chaotic flows and offer explanations for the numerical observations in Figs. 1(a)–1(d) and 2(a)–2(d). For a general chaotic rotation, the phase variable $\phi(t)$ obeys the following equation:

$$\omega(t)=\frac{d\phi}{dt}=\omega_0+F(r,\phi,\mathbf{x}),\qquad(3)$$

where $\mathbf{x}$ is the subset of dynamical variables that are not explicitly utilized in defining the rotation, and F is a nonlinear function characterizing the derivation of the rotation from that of a harmonic oscillator of frequency ω_0. Taking the time derivative of Eq. (3) yields

$$\frac{d\omega}{dt}=\frac{\partial F}{\partial\phi}\omega+\frac{\partial F}{\partial r}\frac{dr}{dt}+\frac{\partial F}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{dt}.\qquad(4)$$
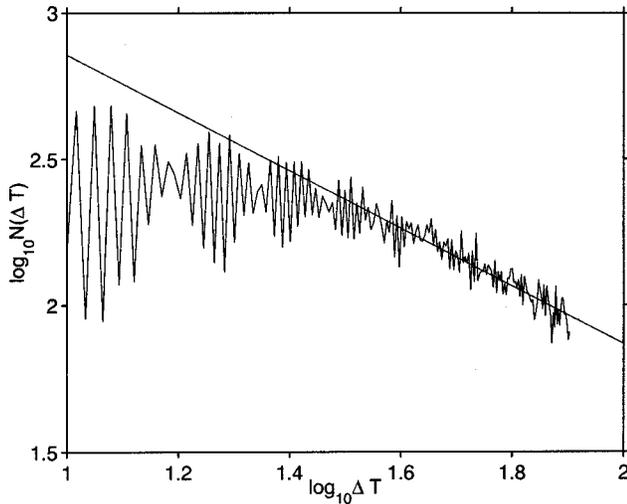
FIG. 3. Algebraic distribution of the intervals between distinct frequency bursts in the Lorenz system [Fig. 2(d)].

FIG. 4. Broadening of the ''off'' state due to symmetry-breaking in the randomly driven logistic map.

Writing $\partial F/\partial \phi \equiv \alpha n_1(t)$ and $(\partial F/\partial r)(dr/dt)+(\partial F/\partial \mathbf{x})\cdot(d\mathbf{x}/dt) \equiv \beta n_2(t)$, where α and β are constants, $n_1(t)$ and $n_2(t)$ are chaotic processes, we obtain $d\omega/dt = \alpha n_1(t)\omega + \beta n_2(t)$. For typical systems such as the Rössler oscillator, the phase angle $\phi(t)$ is a fast variable of time t while ω changes slowly most of the time. Since $n_1(t)$ depends on $\phi(t)$, it is reasonable to assume that the time scales of $n_1(t)$ and $\omega(t)$ are different. Approximately, $n_1(t)$ can be treated as a random process that is independent of $\omega(t)$. Taking the time average of $d\omega/dt$ then yields $\alpha \overline{n_1(t)}\omega_0 + \overline{\beta n_2(t)} = 0$, where $\overline{n_1(t)}$ and $\overline{n_2(t)}$ are the time averages of $n_1(t)$ and $n_2(t)$, respectively. The apparent solution is $\overline{n_1(t)} = \overline{n_2(t)} = 0$ [the other solution $\overline{n_1(t)}/\overline{n_2(t)} = -\beta/(\alpha\omega_0)$] will typically impose an additional constraint on the constants ω_0, α, and β and, consequently, on the original chaotic flow, which is then nonphysical]. We see that $n_1(t)$ and $n_2(t)$ can be treated as random processes with approximately zero mean, which is also consistent with the physical requirement for a chaotic attractor that its rotational frequency not increase or decrease indefinitely. We write $\alpha n_1(t) = h_\perp + \alpha n_0(t)$, where $\overline{n_0(t)} \equiv 0$ and the constant $h_\perp$ is approximately zero. The reason to introduce the quantity $h_\perp$ is to make an analog with the typical system setting for studying on-off intermittency under influence of noise, in which $h_\perp$ is the transverse Lyapunov exponent defined with respect to the invariant manifold, or the ''off'' state [4,6]. Equation (4) is thus completely analogous to, say, the model system for on-off intermittency treated in Ref. [6], and we thus expect to see an intermittent behavior in the instantaneous frequency of a chaotic flow. In particular, say the chaotic variables $n_0(t)$ and $n_2(t)$ have a characteristic time scale τ so that they can be considered as random for $t \gtrsim \tau$. Since $h_\perp$ is essentially zero, we expect the typical time between bursts to be larger than τ and, hence, intermittent bursts can be observed in large time scales and are rare, as shown in Fig. 2(d), where we see that there are approximately 16 distinct bursts for $0 \leq t \leq 600$. Thus, we have $\tau \lesssim 30$ for the Lorenz chaotic attractor. Figure 3 shows, on a logarithmic scale, the statistical distribution of the time intervals between bursts for the Lorenz attractor, where 10^5 bursts are accumu-
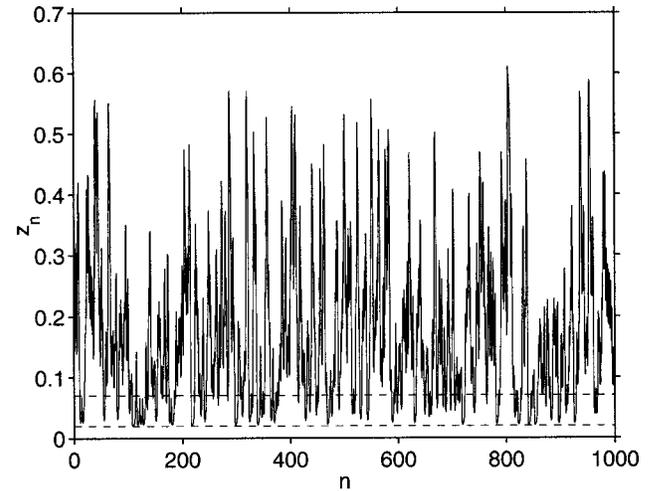
lated to generate the histogram. The distribution shows a clear algebraic behavior, a characteristic feature of on-off intermittency under random noise [4,6]. The exponential behavior expected at large times in the presence of noise or symmetry-breaking [4,10] is, however, apparently not resolved in our numerical experiments.

While the additive term $\beta n_2(t)$ can be regarded as random in large time scales, it has a strong correlation in small time scales, so does the modulation term $\alpha n_0(t)$. Thus, in small time scales, the term $\beta n_2(t)$ can no longer be regarded as random noise. In fact, it is now a *symmetry-breaking* term with respect to the equation: $d\omega/dt = [h_\perp + \alpha n_0(t)]\omega$, which possesses an invariant subspace. As a result, we expect the ''off'' state to be broadened, as can be seen from Fig. 2(d). To better understand the broadening behavior, we consider the following symmetry broken on-off intermittent map: $z_{n+1} = f(z_n) = a x_n z_n(1-z_n) + \epsilon$, where x_n is a random variable uniformly distributed in $[0,1]$ and ϵ is the symmetry-breaking parameter that destroys the invariant subspace $z = 0$. Figure 4 shows an on-off intermittent time series z_n for $a = 2.5$ and $\epsilon = 0.02$. We observe that z_n never goes below the line $z = \epsilon$ so that the ''off'' state is broadened from $z_{\text{off}} = 0$ to $z_{\text{off}} \lesssim \epsilon$. In addition, there is a high probability for the signal to be in the interval $[\epsilon, (1+\alpha)\epsilon]$ (indicated by the two horizontal lines). A detailed analysis of the effect of symmetry breaking on on-off intermittency can be found in Ref. [10].

The Rössler and Lorenz systems that we utilized to demonstrate on-off intermittency in chaotic rotations have the feature that proper rotations can be defined, either by using the original dynamical variables (Fig. 1) or their combination (Fig. 2). What about chaotic flows for which no apparent proper rotation can be defined? Here we wish to suggest our approach to search for intermittent behavior in rotations of such chaotic flows. Given a dynamical variable $x(t)$ from a chaotic flow, we can decompose it into a few number of modes, for which proper rotations can be defined, by using the empirical decomposition procedure [11,3]. The basic idea is that, for a time series coming from a proper rotation, the number of maxima and minima is equal to the number of

zeros in a given large time interval [11]. Thus, an arbitrary time series can be considered as a combination of a number of proper rotations on distinct time scales. We have studied systems such as the Chua's circuit model [12] and found on-off intermittencies in the proper rotations embedded in the underlying chaotic flow.

In summary, we have discovered an intermittent behavior associated with rotations arising in chaotic flows. Our analyses and numerical computations suggest that the intermittent behavior is analogous to on-off intermittency. Besides pro-

viding new insights into the fundamental organization of chaotic flows in terms of rotations [13], our results also imply that on-off intermittency may be more pervasive than previously thought, as its existence in chaotic flows apparently does not rely on symmetry in the system equations.

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[9] In general, there is no guarantee that a chaotic flow possesses a unique center of rotation, although some well studied chaotic systems such as the Rössler oscillator do. If there is no apparent unique center of rotation, a proper change of variable may lead to one, as illustrated by the example with the Lorenz oscillator. Generally, one can make use of the procedure described in Ref. [11] to decompose a chaotic flow into distinct modes, each with a unique center of rotation.

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[13] A widely used technique in the study of chaotic flows is the Poincaré surface-of-section technique. On a Poincaré surface of section, the dynamics can be described by a discrete map whose phase-space dimension is one less than that of the original continuous flow. Chaotic flows can then be understood based on concepts that are convenient for maps such as unstable periodic orbits. The sectioning technique, however, suffers a fundamental drawback: the discrete map produced by it contains no information about the phase of the underlying flow. The framework laid out in this paper, which is based on the idea that chaos is organized around rotations, may thus provide additional insights about chaotic systems which are not revealed by Poincaré maps.